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Clebsch–Gordan products and extended integrity bases of crystallographic double point groups

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Abstract. Clebsch–Gordan products are given for representative double point groups. With the use of a previously described algorithm, the extended integrity bases of irreducible matrix groups defined by representations of double point groups are derived. The bases can be used as a starting point for the calculation of extended integrity bases of double point groups in any set of variables.

1. Introduction

This work is a continuation of two previous papers (Kopský 1979a, b, hereafter referred to as I and II). In paper I we defined the concept of extended integrity bases (EIBS) for finite groups. Further we proved that such bases are finite in a finite set of variables (the extended Noether's theorem); the proof was based on an algorithm which uses successive Clebsch–Gordan reductions with elimination of reducible covariants and which is also suitable for the actual calculation of EIBS. In II we showed from a practical approach that the EIBS of a given group in various sets of variables are composed from EIBS of irreducible matrix groups defined by reps (irreducible representations) of this group; such bases were given for all matrix groups (up to equivalence) defined by reps of crystal point groups.

In this paper we shall extend the results to the crystallographic double point groups. Let us recall the steps we have to follow: (i) firstly, we choose certain matrix reps of the groups in question, with which we shall associate all procedures; (ii) secondly, we find the Clebsch–Gordan products which give a prescription for multiplication of bases (of covariants); (iii) thirdly, we apply the algorithm to derive the EIBS. Since factor groups $G^{(d)}/C_1^{(d)}$ are isomorphic to point groups G for which the EIBS have already been given, it will be sufficient to work only on those reps of $G^{(d)}$ which are not engendered by reps of G (double-valued or projective reps of G). The EIBS of G will be used to simplify the work in $G^{(d)}$. Except for the four-dimensional rep of the group $O^{(d)}$, which generates 19 invariants and 272 covariants and shall therefore be omitted for space reasons, we shall give the EIBS for all irreducible matrix groups defined by reps of crystal double point groups.

Another approach to the calculation of EIBS has been recently reported by Patera *et al* (1978). This and our approach have already been confronted in paper I, to which we refer for details; it is based on the use of generalised Molien series (McLellan 1974) which give a qualitative description of the structure of EIBS, namely, the numbers and degrees of irreducible invariants and covariants and the division of invariants into the

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denominator (free) and numerator (transient) invariants. The EIBs themselves are calculated simply by inspection. We have, in preparation of the present paper, used Molien series to determine the structure of EIBs for cubic groups. These data are not given here, because they may be found in a work by Desmier and Sharp (1979)†, who also calculate the EIBs for double point groups, including the non-crystallographic ones. Our choice of matrix ireps and of EIBs differs from that of Desmier and Sharp (1979) in several cases, and in addition we give the tables of Clebsch–Gordan products from which the calculations can be followed. For the sake of brevity we do not repeat here the bibliography on the subject of EIBs, the most important part of which, we believe, is given by Patera *et al* (1978) and in papers I and II.

2. Double point groups and their representations

2.1. Even and odd representations

The relation between ordinary (crystal) proper rotation groups and corresponding double point groups is given by the homomorphism $\phi: \mathrm{SU}(2) \to \mathrm{SO}(3)$ of the two-dimensional unitary group SU(2) onto the proper rotation group SO(3) for which SU(2) is the representative (covering) group (see e.g. Janssen 1973). Accordingly, to each subgroup $G \subset \mathrm{SO}(3)$ there corresponds a subgroup $G^{(\mathrm{d})} \subset \mathrm{SU}(2)$ so that $\phi G^{(\mathrm{d})} = G$; G is here the proper rotation group, $G^{(\mathrm{d})}$ its double group. The kernel of the homomorphism ϕ is the trivial double point group $\mathrm{C}_1^{(\mathrm{d})} = \ker \phi$, and the ordinary point group G can be considered as the factor group $G^{(\mathrm{d})}/\mathrm{C}_1^{(\mathrm{d})}$. The classes of conjugate elements of $G^{(\mathrm{d})}$ and hence the numbers of reps of $G^{(\mathrm{d})}$ are related to those of G by the Opechowski rules (Opechowski 1940); the reps are well known (see e.g. Bradley and Cracknell 1972).

The reps of $G^{(\mathrm{d})}$ can be divided into two sets: (i) reps engendered by reps of G—the ordinary or single-valued reps of G, which correspond to states with integer spin (momentum) in quantum mechanics; (ii) the remaining reps of $G^{(\mathrm{d})}$—the double-valued reps of G, which correspond to states with half-odd-integer spin. The latter are in fact the projective reps of G; for cyclic groups and for the complete group D_3 they can be transformed by a gauge transformation into the ordinary reps of G. The Schur multiplicator of groups D_2, D_3, D_6, T and O is C_2 (or, in this case, rather $\mathrm{C}_1^{(\mathrm{d})}$), and hence the corresponding double point groups are representative groups, the double-valued reps being projective reps non-similar to the ordinary ones. It should be noted, however, that the double point groups do not exhaust all representative groups (Boyle and Green 1978).

We shall simply call the reps (i) even and the reps (ii) odd, referring to their property that powers and products of even reps are again even, while even powers and products of odd reps are even reps, and odd powers and products of odd reps are odd reps. Since polynomial algebras generated by even reps were considered in II, it is sufficient now to consider only the polynomials in variables belonging to odd reps.

2.2. Representative double point groups and choice of reps

As was said in the Introduction, we shall consider only the EIBs of irreducible matrix

† We received by courtesy of the authors, a preprint of this paper shortly before submission of the present study.

groups defined by reps of double point groups. All such matrix groups defined by odd reps are defined by odd reps of double point groups corresponding to proper rotation groups. Indeed, the non-centrosymmetrical double point groups are isomorphic to these; the centrosymmetrical double point groups provide new odd reps but no new matrix groups. This can be easily checked as a consequence of the fact that each odd rep contains a negative unit matrix. Of the remaining 11 double point groups we shall further eliminate the cyclic groups. The $C_1^{(d)}$, $C_2^{(d)}$ and $C_3^{(d)}$ are isomorphic to C_2, C_4 and C_6 respectively, and for these groups we have already given more exhaustive information in the form of the typical EIBS (Kopský 1975). To transpose the results to the double point groups, it is sufficient only to interpret the variables properly. Though the groups C_8 and C_{12}, which are isomorphic to $C_4^{(d)}$ and $C_6^{(d)}$ respectively, are not included in this work, the procedure of finding the typical integrity bases for them is simple, and bases for cyclic groups have been given by Patera *et al* (1978). We shall therefore consider only the groups $D_2^{(d)}$, $D_3^{(d)}$, $D_4^{(d)}$, $D_6^{(d)}$, $T^{(d)}$ and $O^{(d)}$.

As before we shall use the following convention in defining the reps and the variables. The matrix rep $\Gamma_{0\alpha}(G)$: $g \to D^{(\alpha)}(g)$ defines operation of the group G on a certain typical $\chi_\alpha(G)$-module L_α with basis $\{e_{\alpha i}\}$ by $ge_{\alpha i} = D_{ji}^{(\alpha)}(g)e_{\alpha j}$, while the adjoint basis $\{x_{\alpha i}\}$ of an adjoint $\chi_\alpha^*(G)$-module $\tilde{L}_\alpha$ transforms by adjoint (transposed and reciprocal; in our case conjugate complex because we use unitary reps) matrices: $gx_{\alpha i} = \tilde{D}_{ji}^{(\alpha)}(g)x_{\alpha j}$. We shall continue to use the even reps defined in the work on Clebsch–Gordan products for ordinary point groups (Kopský 1976) as well as the same variables. In defining the odd reps we follow mainly the choice of matrices by Bradley and Cracknell (1972). One of these reps for the groups considered is the half-spin rep defined by matrices

$$D^{1/2}(g) = \pm\begin{pmatrix} e^{i(\alpha+\gamma)/2}\cos\frac{1}{2}\beta & e^{i(-\alpha+\gamma)/2}\sin\frac{1}{2}\beta \\ -e^{i(\alpha-\gamma)/2}\sin\frac{1}{2}\beta & e^{-i(\alpha+\gamma)/2}\cos\frac{1}{2}\beta \end{pmatrix}$$

where $g \in SO(3)$ and α, β, γ are its Euler angles as defined by Bradley and Cracknell (1972). The other two-dimensional reps are defined as products of this rep with suitable one-dimensional even reps. The four-dimensional rep of $O^{(d)}$ is chosen on the grounds of compatibility with reps of $T^{(d)}$. To distinguish clearly the variables to odd reps we use for them the letters ϕ and ψ with appropriate numerical labels.

3. Presentation of results

All specifications and results of calculations are, for convenience, collected in the Appendix. There we specify first for each group its odd reps by giving matrices of group generators. The even reps and corresponding typical variables have already been specified (Kopský 1976). To avoid defining relations of groups we represent the generators by specifically oriented proper rotations.

Further, we give the tables of Clebsch–Gordan products in the way we have used before; i.e. the leading row of the table lists the reps and the notation for the variables; the bilinear covariants are listed in columns below the corresponding reps; the trivial products with invariant x_1 are omitted. Inspection is sufficient for the determination of the Clebsch–Gordan products in most cases. The treatment of more complicated cases as they appear in cubic groups will be illustrated in § 4.1. We also give only the even–odd and odd–odd products, because the even–even products have already been given (Kopský 1976); this completes the tables of Clebsch–Gordan products for double point groups.

Finally we give the EIBs for matrix groups defined by odd reps; those for even reps were given in II and by Patera *et al* (1978). The same algorithm has been used as in II; since its mechanical application leads to unnecessarily complicated covariants in cubic groups we have used our knowledge of the EIBs for even reps (see § 4.2). A spectacular presentation of EIBs in the form of tables analogous to those used in II is avoided here, because such tables will, in the case of odd reps, be extremely complicated. Instead we give the irreducible covariants in a successive list, using notation similar to that employed by Patera *et al* (1978). The denominator invariants are denoted by I, numerator invariants by E, covariants to one-dimensional reps by $E^{(\alpha)}$—these are the so-called relative invariants (Burnside 1955)—and many-dimensional covariants by $\boldsymbol{p}^{(\alpha)}$. The first number in parentheses is the label of variables in which the polynomials are considered, the second is the degree. The label of variables is dropped in the actual expression of the covariant; the many-dimensional covariants are written as row vectors. Within the same group we often shorten the list by comparing a covariant with a previously given one; the equations refer, of course, only to the functional form of these covariants. A conjugate complex $\boldsymbol{p}^{(\alpha)*}$ means $\boldsymbol{p}^{(\alpha)}$ with conjugate complex coefficients, not variables. To distinguish the denominator and numerator invariants we have to check whether the invariants obtained are algebraically independent or not. In the latter case, the corresponding syzygy is given.

4. Illustration of calculations

4.1. Clebsch–Gordan products for cubic groups

A suitable simplification of the calculation is provided by partially classifying the transformation properties of typical variables with respect to one suitable generator. In group $\mathrm{T}^{(\mathrm{d})}$, for example, we use the generator 2_z with eigenvalues 1, −1, −i, i and eigenfunctions as shown in the table.

2_z eigenvalues	2_z eigenfunctions
1	x_1, x_2, x_3, z_4
−1	x_4, y_4
−i	ψ_5, ψ_6, ψ_7
i	ϕ_5, ϕ_6, ϕ_7

It is now easy to classify any product of variables with respect to 2_z. It follows immediately, for example, that $x_2(\phi_7, \psi_7)$ or $x_3(\phi_6, \psi_6)$ are $\Gamma_5^{(1)}$-covariants. Let us further consider the product of (x_4, y_4, z_4) with (ϕ_5, ψ_5). From the characters one finds easily that $\Gamma_4 \otimes \Gamma_5 = \Gamma_5 \oplus \Gamma_6 \oplus \Gamma_7$. From the table above we see at once that $x_4\psi_5$, $y_4\psi_5$ and $z_4\phi_5$ combine into ϕ_5, ϕ_6 or ϕ_7, while $x_4\phi_5$, $y_4\phi_5$ and $z_4\psi_5$ combine into ψ_5, ψ_6 or ψ_7. Now we form linear combinations:

$$\Phi = ax_4\psi_5 + by_4\psi_5 + cz_4\phi_5$$

and

$$\Psi = a'x_4\phi_5 + b'y_4\phi_5 + c'z_4\psi_5,$$

with indeterminate coefficients. To get $(\Phi, \Psi) \approx (\phi_5, \psi_5)$ we determine the coefficients from

$$3\Phi = \tfrac{1}{2}(1-\mathrm{i})(\Phi - \Psi), \qquad 3\Psi = \tfrac{1}{2}(1+\mathrm{i})(\Phi + \Psi).$$

Analogously we get $(\Phi, \Psi) \approx (\phi_6, \psi_6)$ or (ϕ_7, ψ_7) adding factors $-\omega$ or ω^2 respectively into the latter equations.

The table of Clebsch–Gordan products must be self-consistent with respect to mutual substitutions of covariants. This provides many simplifications in its construction as well as a final check. By self-consistence of the table we mean the following. As concerns transformation properties, it is irrelevant whether we consider linear or bilinear covariants, so that we can treat, for example, $x_2(\phi_5, \psi_5)$ as (ϕ_6, ψ_6). Then, if we take, say, the invariant $\phi_5\psi_5 - \psi_5\phi_5$ and multiply it by x_2, we certainly get the Γ_2-covariant, which in view of what has been said above can be written as $\phi_5\psi_6 - \psi_5\phi_6$. Analogous relations hold also in more complicated cases.

4.2. Extended integrity bases of cubic groups

A mechanical application of the algorithm leads, for cubic groups, to rather complicated covariants. It is, however, easy to see that the whole algebra of even-degree polynomials, say in (ϕ_5, ψ_5) in group $T^{(d)}$, must be generated by the least even-degree covariant, which is of the form $(\phi^2 - \psi^2, -\mathrm{i}(\phi^2 + \psi^2), 2\phi\psi)$. This is also, more generally, the $D^{(1)}$-vector of SO(3) in components of the $D^{(1/2)}$-spinor of SU(2). This quadratic covariant is identified with the $\Gamma_4^{(1)}$-covariant (x_4, y_4, z_4) in $T^{(d)}$. The EIB of $\Gamma_4^{(1)}$ has been given in II; there the components x_4, y_4, z_4 were independent variables, while now they are related by $x_4^2 + y_4^2 + z_4^2 = 0$. It is easy to see that we can get the irreducible even-degree covariants in (ϕ_5, ψ_5) by substituting the second-degree covariant into the elements of the integrity basis in (x_4, y_4, z_4). Due to the relation between these components, some terms in the EIB may vanish, but no new terms will appear, since if they are irreducible in (ϕ_5, ψ_5) they will be irreducible also in (x_4, y_4, z_4). The determination of the even-degree irreducible covariants is thus very simple. From them we can then construct the odd-degree covariants with the use of Clebsch–Gordan products. Analogously we can proceed with other variables.

One-dimensional covariants connect the many-dimensional ones and enable us to find useful relations which are given in the Appendix in the relevant places. Thus, for example, $x_2(\phi_6, \psi_6)$ in $O^{(d)}$ is the (ϕ_7, ψ_7) covariant. Since $x_2^{2n} \approx x_1$ and $x_2^{2n+1} \approx x_2$, the even-degree covariants in (ϕ_6, ψ_6) and (ϕ_7, ψ_7) must coincide, while odd-degree covariants have different labels 6 or 7.

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Appendix. Clebsch–Gordan products and extended integrity bases of polynomial algebras generated by odd reps of double point groups $\mathbf{D}_2^{(d)}$, $\mathbf{D}_3^{(d)}$, $\mathbf{D}_4^{(d)}$, $\mathbf{D}_6^{(d)}$, $\mathbf{T}^{(d)}$ and $\mathbf{O}^{(d)}$.

Abbreviations: $\omega = a + \mathrm{i}b = \exp(2\pi\mathrm{i}/6)$, $a = 1/2$, $b = \sqrt{3}/2$; $\theta = \exp(2\pi\mathrm{i}/8) = (1+\mathrm{i})/\sqrt{2}$; $\epsilon = \exp(2\pi\mathrm{i}/12) = b + \mathrm{i}a$.

Group $D_2^{(d)}$-$2_x2_y2_z^{(d)}$—*quaternion group*

Generators: $D_5^{(1)}(2_z)=\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $D_5^{(1)}(2_x)=\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$.

Clebsch–Gordan products

$\Gamma_1(x_1)$	$\Gamma_2(x_2)$	$\Gamma_3(x_3)$	$\Gamma_4(x_4)$	$\Gamma_5^{(1)}(\phi_5, \psi_5)$
$\phi_5\psi_5-\psi_5\phi_5$	$\phi_5\psi_5+\psi_5\phi_5$	$\phi_5^2-\psi_5^2$	$\phi_5^2+\psi_5^2$	$x_2(\phi_5, -\psi_5)$ $x_3(\psi_5, \phi_5)$ $x_4(\psi_5, -\phi_5)$

Extended integrity basis

rep $\Gamma_5^{(1)}(\phi_5, \psi_5)$:

$\Gamma_1(x_1)$: $I_a(5,4)=\phi^2\psi^2$, $I_b(5,4)=\phi^4+\psi^4$, $E(5,6)=\phi\psi(\phi^4-\psi^4)$.
Syzygy: $E^2=I_aI_b^2-4I_a^3$.

$\Gamma_2(x_2)$: $E^{(2)}(5,2)=\phi\psi$, $E^{(2)}(5,4)=\phi^4-\psi^4$.

$\Gamma_3(x_3)$: $E^{(3)}(5,2)=\phi^2-\psi^2$, $E^{(3)}(5,4)=\phi\psi(\phi^2+\psi^2)$.

$\Gamma_4(x_4)$: $E^{(4)}(5,2)=\phi^2+\psi^2$, $E^{(4)}(5,4)=\phi\psi(\phi^2-\psi^2)$.

$\Gamma_5^{(1)}(\phi_5, \psi_5)$: $\boldsymbol{p}^{(5)}(5,1)=(\phi, \psi)$, $\boldsymbol{p}_a^{(5)}(5,3)=(\psi^3, -\phi^3)$, $\boldsymbol{p}_b^{(5)}(5,3)=(\phi^2\psi, -\phi\psi^2)$, $\boldsymbol{p}_a^{(5)}(5,5)=(\phi^5, \psi^5)$, $\boldsymbol{p}_b^{(5)}(5,5)=(\phi\psi^4, \phi^4\psi)$.

Group $D_3^{(d)}$-$3_z2_x^{(d)}$

Generators: $D_4^{(1)}(3_z)=\begin{pmatrix} \omega & 0 \\ 0 & -\omega^2 \end{pmatrix}$, $D_4^{(1)}(2_x)=\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$;

$\chi_5(3_z)=\chi_6(3_z)=-1$, $\chi_5(2_x)=i$, $\chi_6(2_x)=-i$.

Clebsch–Gordan products

$\Gamma_1(x_1)$	$\Gamma_2(x_2)$	$\Gamma_3^{(1)}(\xi_3, \eta_3)$	$\Gamma_4^{(1)}(\phi_4, \psi_4)$	$\Gamma_5(\phi_5)$	$\Gamma_6(\phi_6)$
$\phi_5\phi_6$ $\phi_4\psi_4-\psi_4\phi_4$	$\phi_5^2 \quad \phi_6^2$ $\phi_4\psi_4+\psi_4\phi_4$	$(\phi_4^2, -\psi_4^2)$ $\phi_5(\psi_4, -\phi_4)$ $\phi_6(\psi_4, \phi_4)$	$x_2(\phi_4, -\psi_4)$ $(\xi_3\psi_4, \eta_3\phi_4)$	$x_2\phi_6$ $\xi_3\phi_4+\eta_3\psi_4$	$x_2\phi_5$ $\xi_3\phi_4-\eta_3\psi_4$

Extended integrity bases

reps $\Gamma_5(\phi_5)$ and $\Gamma_6(\phi_6)$: as for cyclic group C_4.

rep $\Gamma_4^{(1)}(\phi_4, \psi_4)$:

$\Gamma_1(x_1)$: $I_0(4,4)=\phi^2\psi^2$, $I_1(4,6)=\phi^6-\psi^6$, $E(5,8)=\phi\psi(\phi^6+\psi^6)$.
Syzygy: $E^2=I_0I_1^2+4I_1^4$.

$\Gamma_2(x_2)$: $E^{(2)}(4,2)=\phi\psi$, $E^{(2)}(4,6)=\phi^6+\psi^6$.

$\Gamma_3^{(1)}(\xi_3, \eta_3)$: $\boldsymbol{p}^{(3)}(4,2) = (\phi^2, -\psi^2)$, $\boldsymbol{p}_a^{(3)}(4,4) = (\phi^3\psi, \phi\psi^3)$, $\boldsymbol{p}_b^{(3)}(4,4) = (\psi^4, \phi^4)$, $\boldsymbol{p}^{(3)}(4,6) = (\phi\psi^5, -\phi^5\psi)$.

$\Gamma_4^{(1)}(\phi_4, \psi_4)$: $\boldsymbol{p}^{(4)}(4,1) = (\phi, \psi)$, $\boldsymbol{p}^{(4)}(4,3) = (\phi^2\psi, -\phi\psi^2)$, $\boldsymbol{p}^{(4)}(4,5) = (\psi^5, \phi^5)$, $\boldsymbol{p}^{(4)}(4,7) = (\phi^7, -\psi^7)$.

$\Gamma_5(\phi_5)$: $E^{(5)}(4,3) = \phi^3 - \psi^3$, $E^{(5)}(4,5) = \phi\psi(\phi^3 + \psi^3)$.

$\Gamma_6(\phi_6)$: $E^{(6)}(4,3) = \phi^3 + \psi^3$, $E^{(6)}(4,5) = \phi\psi(\phi^3 - \psi^3)$.

Group $D_4^{(d)}$–$4_z 2_x 2_{xy}^{(d)}$

Generators: $D_6^{(1)}(4_z) = \begin{pmatrix} \theta & 0 \\ 0 & \theta^* \end{pmatrix}$, $\quad D_6^{(1)}(2_x) = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$;

$D_7^{(1)}(4_z) = \begin{pmatrix} -\theta & 0 \\ 0 & -\theta^* \end{pmatrix}$, $\quad D_7^{(1)}(2_x) = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$.

CG products are given on page 970.

Extended integrity bases

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reps $\Gamma_6^{(1)}(\phi_6, \psi_6)$ and $\Gamma_7^{(1)}(\phi_7, \psi_7)$:

$\boldsymbol{p}^{(\alpha)}(6,k) = \boldsymbol{p}^{(\alpha)}(7,k)$ for $\alpha = 1-5$, $\boldsymbol{p}^{(6)}(6,k) = \boldsymbol{p}^{(7)}(7,k)$, $\boldsymbol{p}^{(6)}(7,k) = \boldsymbol{p}^{(7)}(6,k)$.

$\Gamma_1(x_1)$: $I_0(6,4) = \phi^2\psi^2$, $I_1(6,8) = \phi^8 + \psi^8$, $E(6,10) = \phi\psi(\phi^8 - \psi^8)$.
Syzygy: $E^2 = I_0 I_1^2 - 4I_0^5$.

$\Gamma_2(x_2)$: $E^{(2)}(6,2) = \phi\psi$, $E^{(2)}(6,8) = \phi^8 - \psi^8$.

$\Gamma_3(x_3)$: $E^{(3)}(6,4) = \phi^4 + \psi^4$, $E^{(3)}(6,6) = \phi\psi(\phi^4 - \psi^4)$.

$\Gamma_4(x_4)$: $E^{(4)}(6,4) = \phi^4 - \psi^4$, $E^{(4)}(6,6) = \phi\psi(\phi^4 + \psi^4)$.

$\Gamma_5^{(1)}(\xi_5, \eta_5)$: $\boldsymbol{p}^{(5)}(6,2) = (\phi^2, -\psi^2)$, $\boldsymbol{p}^{(5)}(6,4) = (\phi^3\psi, \phi\psi^3)$, $\boldsymbol{p}^{(5)}(6,6) = (\psi^6, -\phi^6)$, $\boldsymbol{p}^{(5)}(6,8) = (\phi\psi^7, \phi^7\psi)$.

$\Gamma_6^{(1)}(\phi_6, \psi_6)$: $\boldsymbol{p}^{(6)}(6,1) = (\phi, \psi)$, $\boldsymbol{p}^{(6)}(6,3) = (\phi^2\psi, -\phi\psi^2)$, $\boldsymbol{p}^{(6)}(6,7) = (\psi^7, -\phi^7)$, $\boldsymbol{p}^{(6)}(6,9) = (\phi^9, \psi^9)$.

$\Gamma_7^{(1)}(\phi_7, \psi_7)$: $\boldsymbol{p}^{(7)}(6,3) = (\psi^3, -\phi^3)$, $\boldsymbol{p}_a^{(7)}(6,5) = (\phi^5, \psi^5)$, $\boldsymbol{p}_b^{(7)}(6,5) = (\phi\psi^4, \phi^4\psi)$, $\boldsymbol{p}^{(7)}(6,7) = (\phi^6\psi, -\phi\psi^6)$.

Group $D_6^{(d)}$–$6_z 2_x 2_y^{(d)}$

Generators: $D_7^{(1)}(6_z) = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^* \end{pmatrix}$, $\quad D_8^{(1)}(6_z) = \begin{pmatrix} -\epsilon & 0 \\ 0 & -\epsilon^* \end{pmatrix}$, $\quad D_9^{(1)}(6_z) = \begin{pmatrix} -\mathrm{i} & 0 \\ 0 & \mathrm{i} \end{pmatrix}$;

$D_7^{(1)}(2_x) = D_8^{(1)}(2x) = D_9^{(1)}(2_x) = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$.

CG products are given on page 970.

Extended integrity bases

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reps $\Gamma_7^{(1)}(\phi_7, \psi_7)$ and $\Gamma_8^{(1)}(\phi_8, \psi_8)$:

$\boldsymbol{p}^{(\alpha)}(7,k) = \boldsymbol{p}^{(\alpha)}(8,k)$ for $\alpha = 1-6$, $\boldsymbol{p}^{(7)}(7,k) = \boldsymbol{p}^{(8)}(8,k)$, $\boldsymbol{p}^{(7)}(8,k) = \boldsymbol{p}^{(8)}(7,k)$,
$[p_\phi^{(9)}(7,k), p_\psi^{(9)}(7,k)] = [p_\psi^{(9)}(8,k), p_\phi^{(9)}(8,k)]$.

$\Gamma_1(x_1)$: $I_0(7,2) = \phi^2\psi^2$, $I_1(7,12) = \phi^{12} + \psi^{12}$, $E(7,14) = \phi\psi(\phi^{12} - \psi^{12})$.
Syzygy: $E^2 = I_0 I_1^2 - 4I_0^7$.

$\Gamma_2(x_2)$: $E^{(2)}(7,2)=\phi\psi$, $E^{(2)}(7,12)=\phi^{12}-\psi^{12}$.

$\Gamma_3(x_3)$: $E^{(3)}(7,6)=\phi^6-\psi^6$, $E^{(3)}(7,8)=\phi\psi(\phi^6+\psi^6)$.

$\Gamma_4(x_4)$: $E^{(4)}(7,6)=\phi^6+\psi^6$, $E^{(4)}(7,8)=\phi\psi(\phi^6-\psi^6)$.

$\Gamma_5^{(1)}(\xi_5,\eta_5)$: $\boldsymbol{p}^{(5)}(7,4)=(\psi^4,\phi^4)$, $\boldsymbol{p}^{(5)}(7,6)=(\phi\psi^5,-\phi^5\psi)$, $\boldsymbol{p}^{(5)}(7,8)=(\phi^8,\psi^8)$, $\boldsymbol{p}^{(5)}(7,10)=(\phi^9\psi,-\phi\psi^9)$.

$\Gamma_6^{(1)}(\xi_6,\eta_6)$: $\boldsymbol{p}^{(6)}(7,2)=(\phi^2,-\psi^2)$, $\boldsymbol{p}^{(6)}(7,4)=(\phi^3\psi,\phi\psi^3)$, $\boldsymbol{p}^{(6)}(7,10)=(\psi^{10},-\phi^{10})$,
$\boldsymbol{p}^{(6)}(7,12)=(\phi\psi^{11},\phi^{11}\psi)$.

$\Gamma_7^{(1)}(\phi_7,\psi_7)$: $\boldsymbol{p}^{(7)}(7,1)=(\phi,\psi)$, $\boldsymbol{p}^{(7)}(7,3)=(\phi^2\psi,-\phi\psi^2)$, $\boldsymbol{p}^{(7)}(7,11)=(\psi^{11},-\phi^{11})$, $\boldsymbol{p}^{(7)}(7,13)=(\phi^{13},\psi^{13})$.

$\Gamma_8^{(1)}(\phi_8,\psi_8)$: $\boldsymbol{p}^{(8)}(7,5)=(\psi^5,\phi^5)$, $\boldsymbol{p}_a^{(8)}(7,7)=(\phi\psi^6,-\phi^6\psi)$, $\boldsymbol{p}_b^{(8)}(7,7)=(\phi^7,-\psi^7)$, $\boldsymbol{p}^{(8)}(7,9)=(\phi^8\psi,\phi\psi^8)$.

$\Gamma_9^{(1)}(\phi_9,\psi_9)$: $\boldsymbol{p}^{(9)}(7,3)=(\psi^3,-\phi^3)$, $\boldsymbol{p}^{(9)}(7,5)=(\phi\psi^4,\phi^4\psi)$, $\boldsymbol{p}^{(9)}(7,9)=(\phi^9,\psi^9)$,
$\boldsymbol{p}^{(9)}(7,11)=(\phi^{10}\psi,-\phi\psi^{10})$.

rep $\Gamma_9^{(1)}(\phi_9,\psi_9)$ generates the same extended integrity basis as rep $\Gamma_5^{(1)}(\phi_5,\psi_5)$ of $D_2^{(d)}$.

Group $T^{(d)}$–$23^{(d)}$

Generators: $D_5^{(1)}(3)=\dfrac{1+i}{2}\begin{pmatrix}1 & -i\\ -1 & -i\end{pmatrix}$, $D_5^{(1)}(2_z)=\begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix}$;

$$D_6^{(1)}(3)=\frac{\omega^2(1+i)}{2}\begin{pmatrix}1 & -i\\ -1 & -i\end{pmatrix},\qquad D_6^{(1)}(2_z)=\begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix};$$

$$D_7^{(1)}(3)=\frac{-\omega(1+i)}{2}\begin{pmatrix}1 & -i\\ -1 & -i\end{pmatrix},\qquad D_7^{(1)}(2_z)=\begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix}.$$

CG products are given on page 971.

Extended integrity bases

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rep $\Gamma_5^{(1)}(\phi_5,\psi_5)$:

$\Gamma_1(x_1)$: $I_0(5,6)=\phi\psi(\phi^4-\psi^4)$, $I_1(5,8)=\phi^8+\psi^8+14\phi^4\psi^4$, $E(5,12)=(\phi^4+\psi^4)(\phi^8+\psi^8-34\phi^4\psi^4)$.
Syzygy: $E^2=I_1^3-108I_0^4$.

$\Gamma_2(x_2)$: $E^{(2)}(5,4)=\phi^4+\psi^4-4ib\phi^2\psi^2$, $E^{(2)}(5,8)=\phi^8+\psi^8-10\phi^4\psi^4+8ib\phi^2\psi^2(\phi^4+\psi^4)$.

$\Gamma_3(x_3)$: $E^{(3)}=E^{(2)*}$.

$\Gamma_4^{(1)}(x_4,y_4,z_4)$: $\boldsymbol{p}^{(4)}(5,2)=[\phi^2-\psi^2,\ -i(\phi^2+\psi^2),\ 2\phi\psi]$, $\boldsymbol{p}^{(4)}(5,4)=[2\phi\psi(\phi^2+\psi^2),\ 2i\phi\psi(\phi^2-\psi^2),$
$\phi^4-\psi^4]$, $\boldsymbol{p}_a^{(4)}(5,6)=[(\phi^2-\psi^2)^3,\ i(\phi^2+\psi^2)^3,\ 8\phi^3\psi^3]$, $\boldsymbol{p}_b^{(4)}(5,6)=[(\phi^2+\psi^2)(\phi^4-\psi^4),\ 4i\phi^2\psi^2(\phi^2+\psi^2),$
$-2\phi\psi(\phi^2-\psi^2)^2]$, $\boldsymbol{p}^{(4)}(5,8)=[2\phi\psi(\phi^2+\psi^2)^3,\ -8i\phi^3\psi^3(\phi^2-\psi^2),\ -(\phi^4-\psi^4)(\phi^2-\psi^2)^2]$,
$\boldsymbol{p}^{(4)}(5,10)=[(\phi^4-\psi^4)(\phi^2+\psi^2)^3,\ -16i\phi^4\psi^4(\phi^2+\psi^2),\ 2\phi\psi(\phi^2-\psi^2)^4]$.

$\Gamma_5^{(1)}(\phi_5,\psi_5)$: $\boldsymbol{p}^{(5)}(5,1)=(\phi,\psi)$, $\boldsymbol{p}^{(5)}(5,5)=(\phi^5-5\phi\psi^4,\ \psi^5-5\phi^4\psi)$, $\boldsymbol{p}^{(5)}(5,7)=(\psi^7+7\phi^4\psi^3,\ -\phi^7-7\phi^3\psi^4)$,
$\boldsymbol{p}^{(5)}(5,11)=(\psi^{11}-22\phi^4\psi^7-11\phi^8\psi^3,\ -\phi^{11}+22\phi^7\psi^4+11\phi^3\psi^8)$.

$\Gamma_6^{(1)}(\phi_6,\psi_6)$: $\boldsymbol{p}^{(6)}(5,3)=(\psi^3-2ib\phi^2\psi,\ -\phi^3+2ib\phi\psi^2)$, $\boldsymbol{p}^{(6)}(5,5)=(\phi^5-4ib\phi^3\psi^2+\phi\psi^4,$
$\psi^5-4ib\phi^2\psi^3+\phi^4\psi)$, $\boldsymbol{p}^{(6)}(5,7)=(\psi^7+6ib\phi^2\psi^5-5\phi^4\psi^3+2ib\phi^6\psi,\ -\phi^7-6ib\phi^5\psi^2+5\phi^3\psi^4-2ib\phi\psi^6)$,
$\boldsymbol{p}^{(6)}(5,9)=(\phi^9+2ib\phi^7\psi^2-7\phi^5\psi^4+14ib\phi^3\psi^6-2\phi\psi^8,\ \psi^9+2ib\phi^2\psi^7+7\phi^4\psi^5+14ib\phi^6\psi^3-2\phi^8\psi)$.

$\Gamma_7^{(1)}(\phi_7,\psi_7)$: $\boldsymbol{p}^{(7)}(5,k)=\boldsymbol{p}^{(6)*}(5,k)$.

reps $\Gamma_6^{(1)}(\phi_6, \psi_6)$ and $\Gamma_7^{(1)}(\phi_7, \psi_7)$:

$\boldsymbol{p}^{(\alpha)}(6, 3k) = \boldsymbol{p}^{(\alpha)}(7, 3k) = \boldsymbol{p}^{(\alpha)}(5, 3k)$ for any α; $\boldsymbol{p}^{(\alpha)}(6, 3k+1) = \boldsymbol{p}^{(\beta)}(5, 3k+1)$,
$\boldsymbol{p}^{(\alpha)}(7, 3k+2) = \boldsymbol{p}^{(\beta)}(5, 3k+2)$, where $\Gamma_\alpha = \Gamma_2\Gamma_\beta$; $\boldsymbol{p}^{(\alpha)}(6, 3k+2) = \boldsymbol{p}^{(\gamma)}(5, 3k+2)$,
$\boldsymbol{p}^{(\alpha)}(7, 3k+1) = \boldsymbol{p}^{(\gamma)}(5, 3k+1)$, where $\Gamma_\alpha = \Gamma_3\Gamma_\gamma$.

$\Gamma_1(x_1)$: $I(6,4) = \phi^4 + \psi^4 + 4ib\phi^2\psi^2 = E^{(3)}(5,4)$, $I(7,4) = I^*(6,4) = E^{(2)}(5,4)$,
$I(6,6) = I(7,6) = I(5,6) = \phi\psi(\phi^4 - \psi^4)$.

$\Gamma_2(x_2)$ and $\Gamma_3(x_3)$: $E^{(3)}(6,4) = \phi^4 + \psi^4 - 4ib\phi^2\psi^2 = E^{(2)}(5,4)$, $E^{(2)}(7,4) = E^{(3)}(5,4)$,
$E^{(2)}(6,8) = \phi^8 + \psi^8 - 10\phi^4\psi^4 - 8ib\phi^2\psi^2(\phi^4+\psi^4) = E^{(3)}(5,8)$, $E^{(3)}(7,8) = E^{(2)}(5,8)$,
$E^{(3)}(6,8) = \phi^8 + \psi^8 - 14\phi^4\psi^4 = E^{(2)}(7,8) = I(5,8)$.

$\Gamma_4^{(1)}(x_4, y_4, z_4)$: $\boldsymbol{p}^{(4)}(6,2) = [-\omega(\phi^2-\psi^2), -i(\phi^2+\psi^2), 2\omega^2\phi\psi]$, $[\omega^2(\phi^2-\psi^2), -i(\phi^2+\psi^2), -2\omega\phi\psi] =$
$\boldsymbol{p}^{(4)}(7,2)$, $\boldsymbol{p}^{(4)}(6,4) = [2\omega^2\phi\psi(\phi^2+\psi^2), 2i\phi\psi(\phi^2-\psi^2), -\omega(\phi^4-\psi^4)]$,
$[2\omega\phi\psi(\phi^2+\psi^2), -2i\phi\psi(\phi^2-\psi^2), -\omega^2(\phi^4-\psi^4)] = \boldsymbol{p}^{(4)}(7,4)$, $\boldsymbol{p}_a^{(4)}(6,6) = \boldsymbol{p}_a^{(4)}(7,6) = \boldsymbol{p}_a^{(4)}(5,6)$,
or $\boldsymbol{p}_b^{(4)}(6,6) = \boldsymbol{p}_b^{(4)}(7,6) = \boldsymbol{p}_b^{(4)}(5,6)$.

$\Gamma_5^{(1)}(\phi_5, \psi_5)$: $\boldsymbol{p}^{(5)}(6,5) = \boldsymbol{p}^{(6)}(5,5)$, $\boldsymbol{p}^{(5)}(6,7) = \boldsymbol{p}^{(7)}(5,7)$, $\boldsymbol{p}^{(5)}(7,5) = \boldsymbol{p}^{(7)}(5,5)$, $\boldsymbol{p}^{(5)}(7,7) = \boldsymbol{p}^{(6)}(5,7)$.

$\Gamma_6^{(1)}(\phi_6, \psi_6)$ and $\Gamma_7^{(1)}(\phi_7, \psi_7)$: $\boldsymbol{p}^{(6)}(6,1) = \boldsymbol{p}^{(7)}(7,1) = \boldsymbol{p}^{(5)}(5,1) = (\phi, \psi)$, $\boldsymbol{p}^{(6)}(6,3) = \boldsymbol{p}^{(6)}(7,3) = \boldsymbol{p}^{(6)}(5,3)$,
$\boldsymbol{p}^{(7)}(6,3) = \boldsymbol{p}^{(7)}(7,3) = \boldsymbol{p}^{(7)}(5,3)$, $\boldsymbol{p}^{(6)}(7,5) = \boldsymbol{p}^{(7)}(6,5) = \boldsymbol{p}^{(5)}(5,5)$.

Group $O^{(d)}$–$432^{(d)}$

Generators: $D_6^{(1)}(4_z) = \begin{pmatrix} \theta & 0 \\ 0 & \theta^* \end{pmatrix}$, $\quad D_6^{(1)}(3) = \frac{1}{\sqrt{2}}\begin{pmatrix} \theta & \theta^* \\ -\theta & \theta^* \end{pmatrix}$;

$$D_7^{(1)}(4_z) = \begin{pmatrix} -\theta & 0 \\ 0 & -\theta^* \end{pmatrix}, \quad D_7^{(1)}(3) = \frac{1}{\sqrt{2}}\begin{pmatrix} \theta & \theta^* \\ -\theta & \theta^* \end{pmatrix};$$

$$D_8^{(1)}(4_z) = \begin{pmatrix} 0 & 0 & \theta & 0 \\ 0 & 0 & 0 & \theta^* \\ \theta & 0 & 0 & 0 \\ 0 & \theta^* & 0 & 0 \end{pmatrix}, \quad D_8^{(1)}(3) = \frac{1}{\sqrt{2}}\begin{pmatrix} \omega^2\begin{pmatrix} \theta & \theta^* \\ -\theta & \theta^* \end{pmatrix} & 0 \\ 0 & -\omega\begin{pmatrix} \theta & \theta^* \\ -\theta & \theta^* \end{pmatrix} \end{pmatrix}.$$

Clebsch–Gordan products

$\Gamma_1(x_1)$	$\Gamma_2(x_2)$	$\Gamma_3^{(1)}(\xi_3, \eta_3)$
$\phi_6\psi_6 - \psi_6\phi_6$	$\phi_6\psi_7 - \psi_6\phi_7$	$(\bar\phi_8\bar\psi_8 - \bar\psi_8\bar\phi_8,\ \phi_8\psi_8 - \psi_8\phi_8)$
$\phi_7\psi_7 - \psi_7\phi_7$		$(\phi_6\psi_8 - \psi_6\phi_8,\ \phi_6\bar\psi_8 - \psi_6\bar\phi_8)$
$\phi_8\bar\psi_8 - \psi_8\bar\phi_8 + \bar\phi_8\psi_8 - \bar\psi_8\phi_8$	$\phi_8\bar\psi_8 - \psi_8\bar\phi_8 - \bar\phi_8\psi_8 + \bar\psi_8\phi_8$	$(\phi_7\psi_8 - \psi_7\phi_8,\ \psi_7\bar\phi_8 - \phi_7\bar\psi_8)$

$\Gamma_4^{(1)}(x_4, y_4, z_4)$
$[\phi_6\phi_7 - \psi_6\psi_7,\ -i(\phi_6\phi_7 + \psi_6\psi_7),\ \phi_6\psi_7 + \psi_6\phi_7]$
$[(\phi_6\phi_8 - \psi_6\psi_8) + \omega(\phi_6\bar\phi_8 - \psi_6\bar\psi_8),\ i\omega(\phi_6\phi_8 + \psi_6\psi_8) + i(\phi_6\bar\phi_8 + \psi_6\bar\psi_8),\ \omega^2(\phi_6\psi_8 + \psi_6\phi_8) - \omega^2(\phi_6\bar\psi_8 + \psi_6\bar\phi_8)]$
$[(\phi_7\phi_8 - \psi_7\psi_8) - \omega(\phi_7\bar\phi_8 - \psi_7\bar\psi_8),\ i\omega(\phi_7\phi_8 + \psi_7\psi_8) - i(\phi_7\bar\phi_8 + \psi_7\bar\psi_8),\ \omega^2(\phi_7\psi_8 + \psi_7\phi_8) + \omega^2(\phi_7\bar\psi_8 + \psi_7\bar\phi_8)]$
$[\phi_8\bar\phi_8 - \bar\phi_8\phi_8 - \psi_8\bar\psi_8 + \bar\psi_8\psi_8,\ -i(\phi_8\bar\phi_8 - \bar\phi_8\phi_8 + \psi_8\bar\psi_8 - \bar\psi_8\psi_8),\ \phi_8\bar\psi_8 + \psi_8\bar\phi_8 - \bar\psi_8\phi_8 - \bar\phi_8\psi_8]$
$[(\phi_8^2 - \psi_8^2) - \omega^2(\bar\phi_8^2 - \bar\psi_8^2),\ -i\omega^2(\phi_8^2 + \psi_8^2) + i(\bar\phi_8^2 + \bar\psi_8^2),\ -\omega(\phi_8\psi_8 + \psi_8\phi_8) + \omega(\bar\phi_8\bar\psi_8 + \bar\psi_8\bar\phi_8)]$

$\Gamma_5^{(1)}(x_5, y_5, z_5)$
$[\phi_6^2-\psi_6^2,\ -\mathrm{i}(\phi_6^2+\psi_6^2),\ \phi_6\psi_6+\psi_6\phi_6]$
$[\phi_7^2-\psi_7^2,\ -\mathrm{i}(\phi_7^2+\psi_7^2),\ \phi_7\psi_7+\psi_7\phi_7]$
$[(\phi_6\phi_8-\psi_6\psi_8)-\omega(\phi_6\bar\phi_8-\psi_6\bar\psi_8),\ \mathrm{i}\omega(\phi_6\phi_8+\psi_6\psi_8)-\mathrm{i}(\phi_6\bar\phi_8+\psi_6\bar\psi_8),\ \omega^2(\phi_6\psi_8+\psi_6\phi_8)+\omega^2(\phi_6\bar\psi_8+\psi_6\bar\phi_8)]$
$[(\phi_7\phi_8-\psi_7\psi_8)+\omega(\phi_7\bar\phi_8-\psi_7\bar\psi_8),\ \mathrm{i}\omega(\phi_7\phi_8+\psi_7\psi_8)+\mathrm{i}(\phi_7\bar\phi_8+\psi_7\bar\psi_8),\ \omega^2(\phi_7\psi_8+\psi_7\phi_8)-\omega^2(\phi_7\bar\psi_8+\psi_7\bar\phi_8)]$
$[\phi_8\bar\phi_8+\bar\phi_8\phi_8-\psi_8\bar\psi_8-\bar\psi_8\psi_8,\ -\mathrm{i}(\phi_8\bar\phi_8+\bar\phi_8\phi_8+\psi_8\bar\psi_8+\bar\psi_8\psi_8),\ \phi_8\bar\psi_8+\psi_8\bar\phi_8+\bar\phi_8\psi_8+\bar\psi_8\phi_8]$
$[(\phi_8^2-\psi_8^2)+\omega^2(\bar\phi_8^2-\bar\psi_8^2),\ -\mathrm{i}\omega^2(\phi_8^2+\psi_8^2)-\mathrm{i}(\bar\phi_8^2+\bar\psi_8^2),\ -\omega(\phi_8\psi_8+\psi_8\phi_8)-\omega(\bar\phi_8\bar\psi_8+\bar\psi_8\bar\phi_8)]$

$\Gamma_6^{(1)}(\phi_6, \psi_6)$	$\Gamma_7^{(1)}(\phi_7, \psi_7)$
$x_2(\phi_7, \psi_7)$	$x_2(\phi_6, \psi_6)$
$(\eta_3\phi_8+\xi_3\bar\phi_8;\ \eta_3\psi_8+\xi_3\bar\psi_8)$	$(\eta_3\phi_8-\xi_3\bar\phi_8,\ \eta_3\psi_8-\xi_3\bar\psi_8)$
$[(x_5+\mathrm{i}y_5)\psi_6-z_5\phi_6,\ (x_5-\mathrm{i}y_5)\phi_6+z_5\psi_6]$	$[(x_4+\mathrm{i}y_4)\psi_6-z_4\phi_6,\ (x_4-\mathrm{i}y_4)\phi_6+z_4\psi_6]$
$[(x_4+\mathrm{i}y_4)\psi_7-z_4\phi_7,\ (x_4-\mathrm{i}y_4)\phi_7+z_4\psi_7]$	$[(x_5+\mathrm{i}y_5)\psi_7-z_5\phi_7,\ (x_5-\mathrm{i}y_5)\phi_7+z_5\psi_7]$
$[(\omega x_5-\mathrm{i}\omega^2y_5)\psi_8-(\omega^2x_5-\mathrm{i}\omega y_5)\bar\psi_8+z_5(\phi_8+\bar\phi_8),\ (\omega x_5+\mathrm{i}\omega^2y_5)\phi_8-(\omega^2x_5+\mathrm{i}\omega y_5)\bar\phi_8-z_5(\psi_8+\bar\psi_8)]$	$[(\omega x_4-\mathrm{i}\omega^2y_4)\psi_8-(\omega^2x_4-\mathrm{i}\omega y_4)\bar\psi_8+z_4(\phi_8+\bar\phi_8),\ (\omega x_4+\mathrm{i}\omega^2y_4)\phi_8-(\omega^2x_4+\mathrm{i}\omega y_4)\bar\phi_8-z_4(\psi_8+\bar\psi_8)]$
$[(\omega x_4-\mathrm{i}\omega^2y_4)\psi_8+(\omega^2x_4-\mathrm{i}\omega y_4)\bar\psi_8+z_4(\phi_8-\bar\phi_8),\ (\omega x_4+\mathrm{i}\omega^2y_4)\phi_8+(\omega^2x_4+\mathrm{i}\omega y_4)\bar\phi_8-z_4(\psi_8-\bar\psi_8)]$	$[(\omega x_5-\mathrm{i}\omega^2y_5)\psi_8+(\omega^2x_5-\mathrm{i}\omega y_5)\bar\psi_8+z_5(\phi_8-\bar\phi_8),\ (\omega x_5+\mathrm{i}\omega^2y_5)\phi_8+(\omega^2x_5+\mathrm{i}\omega y_5)\bar\phi_8-z_5(\psi_8-\bar\psi_8)]$

$\Gamma_8^{(1)}(\phi_8, \psi_8;\ \bar\phi_8, \bar\psi_8)$
$x_2(\phi_8, \psi_8;\ -\bar\phi_8, -\bar\psi_8)$
$(\xi_3\phi_6, \xi_3\psi_6;\ \eta_3\phi_6, \eta_3\psi_6)$
$(\xi_3\phi_7, \xi_3\psi_7;\ -\eta_3\phi_7, -\eta_3\psi_7)$
$(\eta_3\bar\phi_8, \eta_3\bar\psi_8;\ \xi_3\phi_8, \xi_3\bar\phi_8)$
$[(\omega^2x_5-\mathrm{i}\omega y_5)\psi_6-z_5\phi_6,\ (\omega^2x_5+\mathrm{i}\omega y_5)\phi_6+z_5\psi_6;\ -(\omega x_5-\mathrm{i}\omega^2y_5)\psi_6-z_5\phi_6,\ -(\omega x_5+\mathrm{i}\omega^2y_5)\phi_6+z_5\psi_6]$
$[(\omega^2x_4-\mathrm{i}\omega y_4)\psi_7-z_4\phi_7,\ (\omega^2x_4+\mathrm{i}\omega y_4)\phi_7+z_4\psi_7;\ -(\omega x_4-\mathrm{i}\omega^2y_4)\psi_7-z_4\phi_7,\ -(\omega x_4+\mathrm{i}\omega^2y_4)\phi_7+z_4\psi_7]$
$[(\omega^2x_5-\mathrm{i}\omega y_5)\psi_7-z_5\phi_7,\ (\omega^2x_5+\mathrm{i}\omega y_5)\phi_7+z_5\psi_7;\ (\omega x_5-\mathrm{i}\omega^2y_5)\psi_7+z_5\phi_7,\ (\omega x_5+\mathrm{i}\omega^2y_5)\phi_7-z_5\psi_7]$
$[(\omega^2x_4-\mathrm{i}\omega y_4)\psi_6-z_4\phi_6,\ (\omega^2x_4+\mathrm{i}\omega y_4)\phi_6+z_4\psi_6;\ (\omega x_4-\mathrm{i}\omega^2y_4)\psi_6+z_4\phi_6,\ (\omega x_4+\mathrm{i}\omega^2y_4)\phi_6-z_4\psi_6]$
$[(x_5+\mathrm{i}y_5)\psi_8-z_5\phi_8,\ (x_5-\mathrm{i}y_5)\phi_8+z_5\psi_8;\ (x_5+\mathrm{i}y_5)\bar\psi_8-z_5\bar\phi_8,\ (x_5-\mathrm{i}y_5)\bar\phi_8+z_5\bar\psi_8]$
$[(x_4+\mathrm{i}y_4)\psi_8-z_4\phi_8,\ (x_4-\mathrm{i}y_4)\phi_8+z_4\psi_8;\ -(x_4+\mathrm{i}y_4)\bar\psi_8+z_4\bar\phi_8,\ -(x_4-\mathrm{i}y_4)\bar\phi_8-z_4\bar\psi_8]$
$[(\omega x_5-\mathrm{i}\omega^2y_5)\bar\psi_8+z_5\bar\phi_8,\ (\omega x_5+\mathrm{i}\omega^2y_5)\bar\phi_8-z_5\bar\psi_8;\ (-\omega^2x_5+\mathrm{i}\omega y_5)\psi_8+z_5\phi_8,\ (-\omega^2x_5-\mathrm{i}\omega y_5)\phi_8-z_5\psi_8]$
$[(\omega x_4-\mathrm{i}\omega^2y_4)\bar\psi_8+z_4\bar\phi_8,\ (\omega x_4+\mathrm{i}\omega^2y_4)\bar\phi_8-z_4\bar\psi_8;\ (\omega^2x_4-\mathrm{i}\omega y_4)\psi_8-z_4\phi_8,\ (\omega^2x_4+\mathrm{i}\omega y_4)\phi_8+z_4\psi_8]$

Extended integrity bases

reps $\Gamma_6^{(1)}(\phi_6, \psi_6)$ and $\Gamma_7^{(1)}(\phi_7, \psi_7)$:

$\boldsymbol{p}^{(\alpha)}(6, k)=\boldsymbol{p}^{(\alpha)}(7, k)$ for $\alpha=1-5$, $\boldsymbol{p}^{(6)}(6, k)=\boldsymbol{p}^{(7)}(7, k)$, $\boldsymbol{p}^{(7)}(6, k)=\boldsymbol{p}^{(6)}(7, k)$, $[p_\phi^{(8)}(6, k), p_\psi^{(8)}(6, k);\ p_{\bar\phi}^{(8)}(6, k), p_{\bar\psi}^{(8)}(6, k)]=[p_{\bar\phi}^{(8)}(7, k), p_{\bar\psi}^{(8)}(7, k), -p_\phi^{(8)}(7, k), -p_\psi^{(8)}(7, k)]$.

$\Gamma_1(x_1)$: $I_0(6, 8)=\phi^8+\psi^8+14\phi^4\psi^4$, $I_1(6, 12)=\phi^2\psi^2(\phi^4-\psi^4)^2$, $E(6, 18)=\phi\psi(\phi^8-\psi^8)(\phi^8+\psi^8-34\phi^4\psi^4)$. Syzygy: $E^2=I_1I_0^3-108I_1^3$.

$\Gamma_2(x_2)$: $E^{(2)}(6,6) = \phi\psi(\phi^4 - \psi^4)$, $E^{(2)}(6,12) = (\phi^4 + \psi^4)(\phi^8 + \psi^8 - 34\phi^4\psi^4)$.

$R_3^{(1)}(x_3, y_3)$: $\boldsymbol{p}^{(3)}(6,4) = [3\phi^2\psi^2,\ b(\phi^4 + \psi^4)]$, $\boldsymbol{p}^{(3)}(6,8) = [\phi^8 + \psi^8 - 13\phi^4\psi^4,\ 8b\phi^2\psi^2(\phi^4 + \psi^4)]$, $\boldsymbol{p}^{(3)}(6,10) = E^{(2)}(6,6)[b(\phi^4 + \psi^4),\ -3\phi^2\psi^2]$, $\boldsymbol{p}^{(3)}(6,14) = E^{(2)}(6,6)[8b\phi^2\psi^2(\phi^4 + \psi^4),\ -\phi^8 - \psi^8 + 13\phi^4\psi^4]$.

$\Gamma_4^{(1)}(x_4, y_4, z_4)$: $\boldsymbol{p}^{(4)}(6,4) = [2\phi\psi(\phi^2 + \psi^2),\ 2\mathrm{i}\phi\psi(\phi^2 - \psi^2),\ \phi^4 - \psi^4]$, $\boldsymbol{p}^{(4)}(6,6) = [\phi^6 - \psi^6 + 5\phi^2\psi^2(\phi^2 - \psi^2),\ -\mathrm{i}\{\phi^6 + \psi^6 - 5\phi^2\psi^2(\phi^2 + \psi^2)\},\ -4\phi\psi(\phi^4 + \psi^4)]$, $\boldsymbol{p}^{(4)}(6,8) = E^{(2)}(6,6)\boldsymbol{p}^{(5)}(6,2)$, $\boldsymbol{p}^{(4)}(6,10) = [-\phi^{10} + \psi^{10} - 3\phi^2\psi^2(\phi^6 - \psi^6) + 14\phi^4\psi^4(\phi^2 - \psi^2),\ -\mathrm{i}\{\phi^{10} + \psi^{10} - 3\phi^2\psi^2(\phi^6 + \psi^6) - 14\phi^4\psi^4(\phi^2 + \psi^2),\ 16\phi^3\psi^3(\phi^4 + \psi^4)]$, $\boldsymbol{p}^{(4)}(6,12) = E^{(2)}(6,6)\boldsymbol{p}^{(5)}(6,6)$, $\boldsymbol{p}^{(4)}(6,14) = E^{(2)}(6,6)\boldsymbol{p}^{(5)}(6,8)$.

$\Gamma_5^{(1)}(x_5, y_5, z_5)$: $\boldsymbol{p}^{(5)}(6,2) = [\phi^2 - \psi^2,\ -\mathrm{i}(\phi^2 + \psi^2),\ 2\phi\psi]$, $\boldsymbol{p}^{(5)}(6,6) = [(\phi^2 - \psi^2)^3,\ \mathrm{i}(\phi^2 + \psi^2)^3,\ 8\phi^3\psi^3]$, $\boldsymbol{p}^{(5)}(6,8) = [\phi\psi\{\phi^6 + \psi^6 + 7\phi^2\psi^2(\phi^2 + \psi^2)\},\ \mathrm{i}\phi\psi\{\phi^6 - \psi^6 - 7\phi^2\psi^2(\phi^2 - \psi^2)\},\ -\phi^8 + \psi^8]$, $\boldsymbol{p}^{(5)}(6,10) = E^{(2)}(6,6)\boldsymbol{p}^{(4)}(6,4)$, $\boldsymbol{p}^{(5)}(6,12) = E^{(2)}(6,6)\boldsymbol{p}^{(4)}(6,6)$, $\boldsymbol{p}^{(5)}(6,16) = E^{(2)}(6,6)\boldsymbol{p}^{(4)}(6,10)$.

$\Gamma_6^{(1)}(\phi_6, \psi_6)$: $\boldsymbol{p}^{(6)}(6,1) = (\phi, \psi)$, $\boldsymbol{p}^{(6)}(6,7) = (\phi^7 + 7\phi^4\psi^3,\ -\psi^7 - 7\phi^3\psi^4)$, $\boldsymbol{p}^{(6)}(6,11) = E^{(2)}(6,6)\boldsymbol{p}^{(7)}(6,5)$, $\boldsymbol{p}^{(6)}(6,17) = E^{(2)}(6,6)\boldsymbol{p}^{(7)}(6,11)$.

$\Gamma_7^{(1)}(\phi_7, \psi_7)$: $\boldsymbol{p}^{(7)}(6,5) = (\phi^5 - 5\phi\psi^4,\ \psi^5 - \phi^4\psi)$, $\boldsymbol{p}^{(7)}(6,7) = (\phi^6\psi - \phi^2\psi^5,\ -\phi\psi^6 + \phi^5\psi^2)$, $\boldsymbol{p}^{(7)}(6,11) = (\psi^{11} - 22\phi^4\psi^7 - 11\phi^8\psi^3,\ -\phi^{11} + 22\phi^7\psi^4 + 11\phi^3\psi^8)$, $\boldsymbol{p}^{(7)}(6,13) = E^{(2)}(6,6)\boldsymbol{p}^{(6)}(6,7)$.

$\Gamma_8^{(1)}(\phi_8, \psi_8;\ \bar{\phi}_8, \bar{\psi}_8)$: $\boldsymbol{p}^{(8)}(6,3) = (\psi^3 - 2\mathrm{i}b\phi^2\psi,\ -\phi^3 + 2\mathrm{i}b\phi\psi^2;\ -\psi^3 - 2\mathrm{i}b\phi^2\psi,\ \phi^3 + 2\mathrm{i}b\phi\psi^2)$, $\boldsymbol{p}^{(8)}(6,5) = (\phi^5 + \phi\psi^4 - 4\mathrm{i}b\phi^3\psi^2,\ \psi^5 + \phi^4\psi - 4\mathrm{i}b\phi^2\psi^3;\ -\phi^5 - \phi\psi^4 - 4\mathrm{i}b\phi^3\psi^2,\ -\psi^5 - \phi^4\psi - 4\mathrm{i}b\phi^2\psi^3)$, $\boldsymbol{p}^{(8)}(6,7) = (\psi^7 - 5\phi^4\psi^3 + 2\mathrm{i}b(\phi^6\psi + 3\phi^2\psi^5),\ -\phi^7 + 5\phi^3\psi^4 - 2\mathrm{i}b(\phi\psi^6 + 3\phi^5\psi^2);\ \psi^7 - 5\phi^4\psi^3 - 2\mathrm{i}b(\phi^6\psi + 3\phi^2\psi^5),\ -\phi^7 + 5\phi^3\psi^4 + 2\mathrm{i}b(\phi\psi^6 + 3\phi^5\psi^2))$, $\boldsymbol{p}^{(8)}(6,9) = (\phi^9 + 2\mathrm{i}b\phi^7\psi^2 - 7\phi^5\psi^4 + 14\mathrm{i}b\phi^3\psi^6 - 2\phi\psi^8,\ \psi^9 + 2\mathrm{i}b\phi^2\psi^7 - 7\phi^4\psi^5 + 14\mathrm{i}b\phi^6\psi^3 - 2\phi^8\psi;\ \phi^9 - 2\mathrm{i}b\phi^7\psi^2 - 7\phi^5\psi^4 - 14\mathrm{i}b\phi^3\psi^5 - 2\phi\psi^8,\ \psi^9 - 2\mathrm{i}b\phi^2\psi^7 - 7\phi^4\psi^5 - 14\mathrm{i}b\phi^6\psi^3 - 2\phi^4\psi)$, $\boldsymbol{p}^{(8)}(6,11) = E^{(2)}(6,6)\cdot(p_\phi^{(8)}, p_\psi^{(8)};\ -p_\phi^{(8)}, -p_\psi^{(8)})(6,5)$, $\boldsymbol{p}^{(8)}(6,13) = E^{(2)}(6,6)\cdot(p_\phi^{(8)}, p_\psi^{(8)};\ -p_\phi^{(8)}, -p_\psi^{(8)})(6,7)$, $\boldsymbol{p}^{(8)}(6,15) = E^{(2)}(6,6)\cdot(p_\phi^{(8)}, p_\psi^{(8)};\ -p_\phi^{(8)}, -p_\psi^{(8)})(6,9)$.

Group $D_4^{(d)}$–$4_z 2_x 2_{xy}^{(d)}$

Clebsch–Gordan products

$\Gamma_1(x_1)$	$\Gamma_2(x_2)$	$\Gamma_3(x_3)$	$\Gamma_4(x_4)$	$\Gamma_5^{(1)}(\xi_5, \eta_5)$	$\Gamma_6^{(1)}(\phi_6, \psi_6)$	$\Gamma_7^{(1)}(\phi_7, \psi_7)$
$\phi_6\psi_6 - \psi_6\phi_6$ $\phi_7\psi_7 - \psi_7\phi_7$	$\phi_6\psi_6 + \psi_6\phi_6$ $\phi_7\psi_7 + \psi_7\phi_7$	$\phi_6\psi_7 - \psi_6\phi_7$	$\phi_6\psi_7 + \psi_6\phi_7$	$(\phi_6^2, -\psi_6^2)$ $(\phi_7^2, -\psi_7^2)$ $(\psi_6\psi_7, -\phi_6\phi_7)$	$x_2(\phi_6, -\psi_6)$ $x_3(\phi_7, \psi_7)$ $x_4(\phi_7, -\psi_7)$ $(\xi_5\psi_6, \eta_5\phi_6)$ $(\eta_5\psi_7, \xi_5\phi_7)$	$x_2(\phi_7, -\psi_7)$ $x_3(\phi_6, \psi_6)$ $x_4(\phi_6, -\psi_6)$ $(\xi_5\psi_7, \eta_5\phi_7)$ $(\eta_5\psi_6, \xi_5\phi_6)$

Group $D_6^{(d)}$–$6_z 2_x 2_y^{(d)}$

Clebsch–Gordan products

$\Gamma_1(x_1)$	$\Gamma_2(x_2)$	$\Gamma_3(x_3)$	$\Gamma_4(x_4)$	$\Gamma_5^{(1)}(\xi_5, \eta_5)$	$\Gamma_6^{(1)}(\xi_6, \eta_6)$
$\phi_7\psi_7 - \psi_7\phi_7$ $\phi_8\psi_8 - \psi_8\phi_8$ $\phi_9\psi_9 - \psi_9\phi_9$	$\phi_7\psi_7 + \psi_7\phi_7$ $\phi_8\psi_8 + \psi_8\phi_8$ $\phi_9\psi_9 + \psi_9\phi_9$	$\phi_7\psi_8 - \psi_7\phi_8$ $\phi_9^2 - \psi_9^2$	$\phi_7\psi_8 + \psi_7\phi_8$ $\phi_9^2 + \psi_9^2$	$(\phi_7\phi_8, -\psi_7\psi_8)$ $(\psi_7\phi_9, -\phi_7\psi_9)$ $(\psi_8\psi_9, -\phi_8\phi_9)$	$(\phi_7^2, -\psi_7^2)$ $(\phi_8^2, -\psi_8^2)$ $(\psi_7\psi_9, -\phi_7\phi_9)$ $(\psi_8\phi_9, -\phi_8\psi_9)$

$\Gamma_7^{(1)}(\phi_7, \psi_7)$	$\Gamma_8^{(1)}(\phi_8, \psi_8)$	$\Gamma_9^{(1)}(\phi_9, \psi_9)$
$x_2(\phi_7, -\psi_7)$ $x_3(\phi_8, \psi_8)$ $x_4(\phi_8, -\psi_8)$ $(\xi_6\psi_7, \eta_6\phi_7)$ $(\xi_5\psi_8, \eta_5\phi_8)$ $(\eta_5\phi_9, \xi_5\psi_9)$ $(\eta_6\psi_9, \xi_6\phi_9)$	$x_2(\phi_8, -\psi_8)$ $x_3(\phi_7, \psi_7)$ $x_4(\phi_8, -\psi_8)$ $(\xi_5\psi_7, \eta_5\phi_7)$ $(\xi_6\psi_8, \eta_6\phi_8)$ $(\eta_5\psi_9, \xi_5\phi_9)$ $(\eta_6\phi_9, \xi_6\psi_9)$	$x_2(\phi_9, -\psi_9)$ $x_3(\psi_9, \phi_9)$ $x_4(\psi_9, -\phi_9)$ $(\xi_5\phi_7, \eta_5\psi_7)$ $(\eta_6\psi_7, \xi_6\phi_7)$ $(\eta_5\psi_8, \xi_5\phi_8)$ $(\xi_6\phi_8, \eta_6\psi_8)$

Group $T^{(d)}$-$23^{(d)}$

Clebsch–Gordan products

$\Gamma_1(x_1)$	$\Gamma_2(x_2)$	$\Gamma_3(x_3)$	$\Gamma_4^{(1)}(x_4, y_4, z_4)$
$\phi_5\psi_5-\psi_5\phi_5$	$\phi_5\psi_6-\psi_5\phi_6$	$\phi_5\psi_7-\psi_5\phi_7$	$[\phi_5^2-\psi_5^2, -i(\phi_5^2+\psi_5^2), \phi_5\psi_5+\psi_5\phi_5]$
$\phi_6\psi_7-\psi_6\phi_7$	$\phi_7\psi_7-\psi_7\phi_7$	$\phi_6\psi_6-\psi_6\phi_6$	$[-\omega(\phi_6^2-\psi_6^2), -i(\phi_6^2+\psi_6^2), \omega^2(\phi_6\psi_6+\psi_6\phi_6)]$
			$[\omega^2(\phi_7^2-\psi_7^2), -i(\phi_7^2+\psi_7^2), -\omega(\phi_7\psi_7+\psi_7\phi_7)]$
			$[\phi_6\phi_7-\psi_6\psi_7, -i(\phi_6\phi_7+\psi_6\psi_7), \phi_6\psi_7+\psi_6\phi_7]$
			$[-\omega(\phi_5\phi_7-\psi_5\psi_7), -i(\phi_5\phi_7+\psi_5\psi_7), \omega^2(\phi_5\psi_7+\psi_5\phi_7)]$
			$[\omega^2(\phi_5\phi_6-\psi_5\psi_6), -i(\phi_5\phi_6+\psi_5\psi_6), -\omega(\phi_5\psi_6+\psi_5\phi_6)]$

$\Gamma_5^{(1)}(\phi_5, \psi_5)$	$\Gamma_6^{(1)}(\phi_6, \psi_6)$	$\Gamma_7^{(1)}(\phi_7, \psi_7)$
$x_2(\phi_7, \psi_7)$	$x_2(\phi_5, \psi_5)$	$x_2(\phi_6, \psi_6)$
$x_3(\phi_6, \psi_6)$	$x_3(\phi_7, \psi_7)$	$x_3(\phi_5, \psi_5)$
$[(x_4+iy_4)\psi_5-z_4\phi_5, (x_4-iy_4)\phi_5+z_4\psi_5]$	$[(x_4+iy_4)\psi_6-z_4\phi_6, (x_4-iy_4)\phi_6+z_4\psi_6]$	$[(x_4+iy_4)\psi_7-z_4\phi_7, (x_4-iy_4)\phi_7+z_4\psi_7]$
$[(\omega x_4-i\omega^2y_4)\psi_6+z_4\phi_6, (\omega x_4+i\omega^2y_4)\phi_6-z_4\psi_6]$	$[(\omega x_4-i\omega^2y_4)\psi_7+z_4\phi_7, (\omega x_4+i\omega^2y_4)\phi_7-z_4\psi_7]$	$[(\omega x_4-i\omega^2y_4)\psi_5+z_4\phi_5, (\omega x_4+i\omega^2y_4)\phi_5-z_4\psi_5]$
$[(\omega^2x_4-i\omega y_4)\psi_7-z_4\phi_7, (\omega^2x_4+i\omega y_4)\phi_7+z_4\psi_7]$	$[(\omega^2x_4-i\omega y_4)\psi_5-z_4\phi_5, (\omega^2x_4+i\omega y_4)\phi_5+z_4\psi_5]$	$[(\omega^2x_4-i\omega y_4)\psi_6-z_4\phi_6, (\omega^2x_4+i\omega y_4)\phi_6+z_4\psi_6]$

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